Continuidad de Funciones

(*a*) $f(x_0)$ exists

(*b*) $\lim_{x \to x_0} f(x)$ exists

(*c*) $\lim_{x \to x_0} f(x) = f(x_0)$

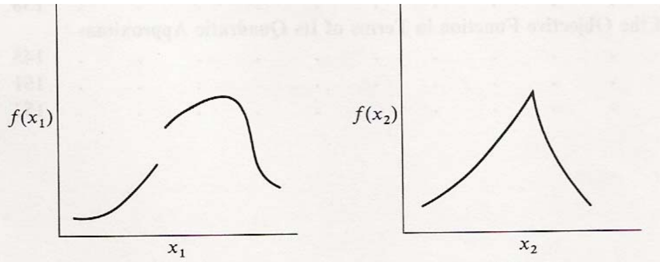

Figure 4.1 Functions with discontinuities in the function and/or derivatives.

EXAMPLE 4.1 ANALYSIS OF FUNCTIONS FOR CONTINUITY

Are the following functions continuous? (*a*) $f(x) = 1/x$; (*b*) $f(x) = \ln\ x$. In each case specify the range of x for which $f(x)$ and $f'(x)$ are continuous.

Solution

(*a*) $f(x) = 1/x$ is continuous except at $x = 0$; $f(0)$ is not defined. $f'(x) = -1/x^2$ is continuous except at $x = 0$.

(*b*) $f(x) = \ln\ x$ is continuous for $x > 0$. For $x \leq 0$, $\ln(x)$ is not defined. As to $f'(x) = 1/x$, see (*a*).

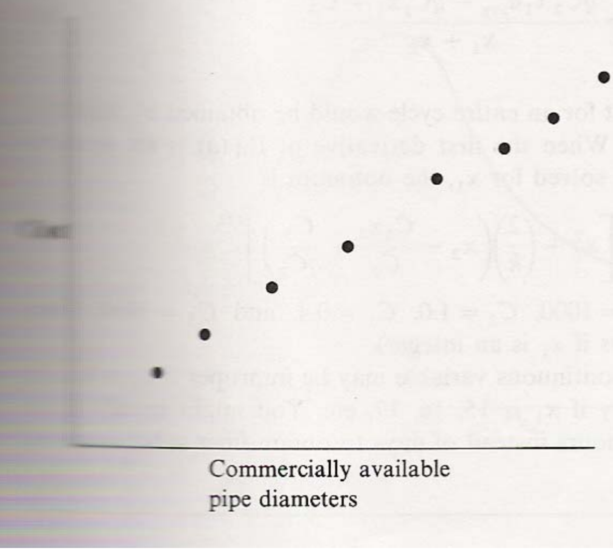

Figure 4.2 Installed pipe cost as a function of diameter.

Funciones Unimodales vs. Funciones Multimodales

$$f(x_1) < f(x_2) < f(x^*) \qquad x_1 < x_2 < x^*$$

$$f(x_4) < f(x_3) < f(x^*) \qquad x^* < x_3 < x_4$$

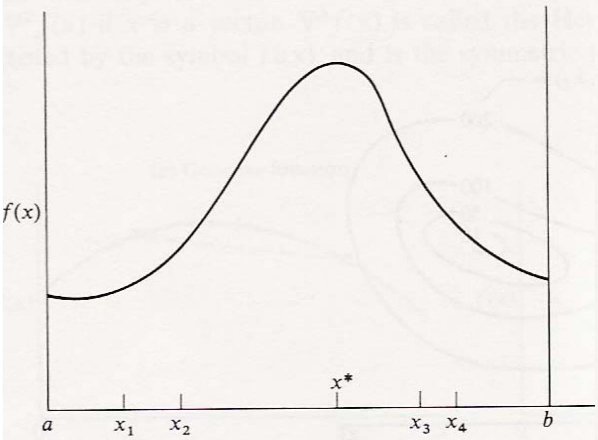

Figure 4.3*a* A unimodal function.

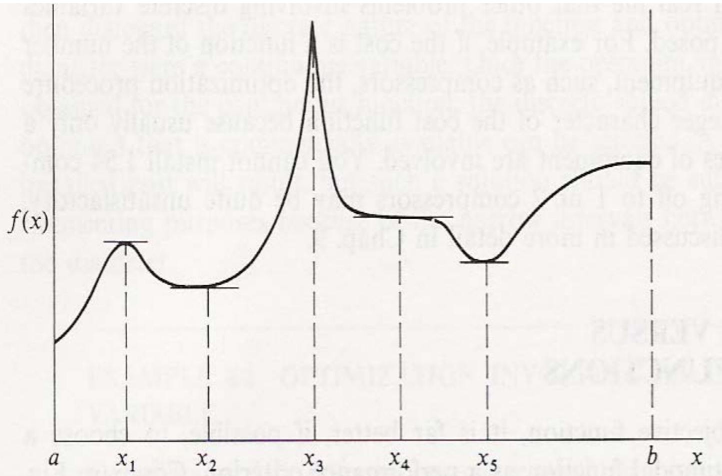

Figure 4.3*b* A multimodal function.

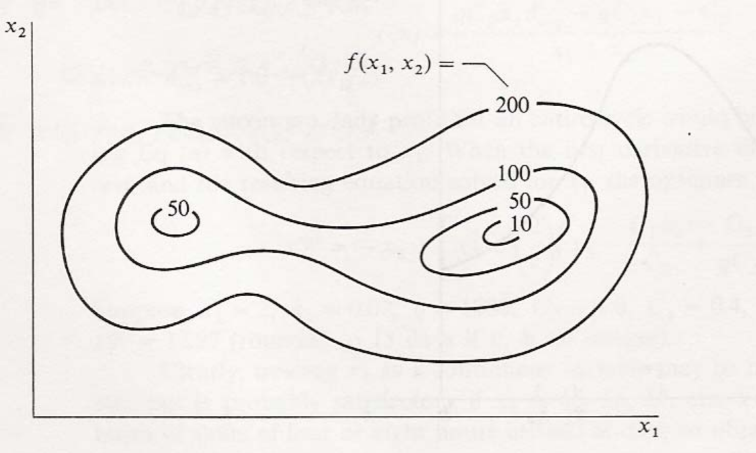

Figure 4.4 Contours of $f(x_1, x_2)$, a multimodal function.

Funciones Cóncavas y Convexas

La determinación de la concavidad o convexidad ayuda a determinar si un óptimo es local o global

Una función se dice cóncava si:

$$f[\theta x_a + (1-\theta)x_b] \geq \theta f(x_a) + (1-\theta) f(x_b)$$

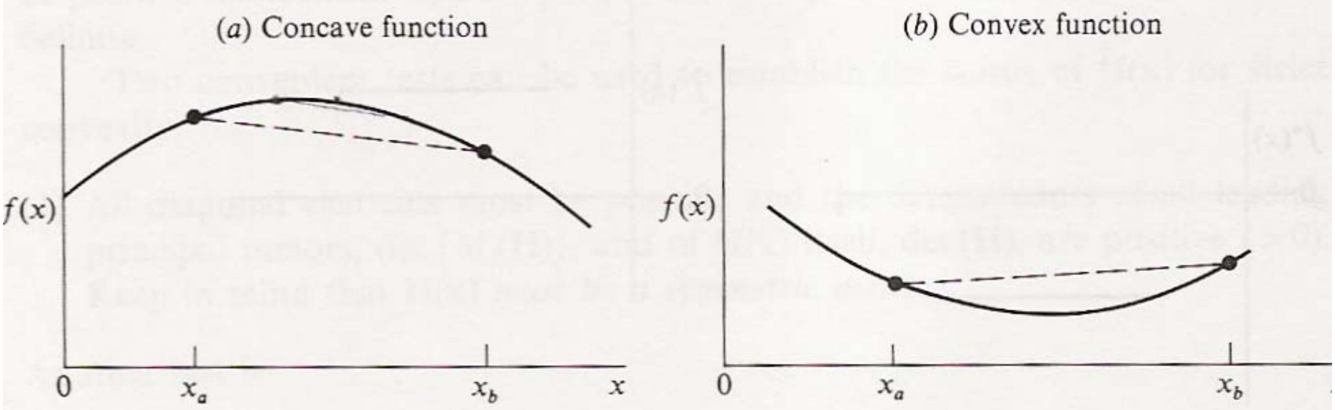

Figure 4.5 Comparison of concave (*a*) and convex (*b*) functions.

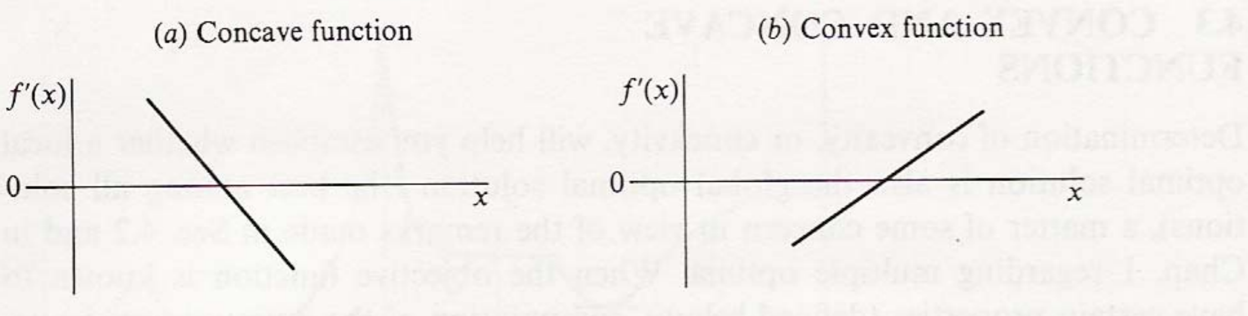

Figure 4.6 Plot of the first derivatives of a quadratic concave and convex function of a single variable.

$$f(\mathbf{x}) = h_{11}x_1^2 + h_{12}x_1x_2 + h_{22}x_2^2$$

$$\mathbf{H}(\mathbf{x}) \equiv \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f(\mathbf{x})}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2h_{11} & h_{12} \\ h_{12} & 2h_{22} \end{bmatrix}$$

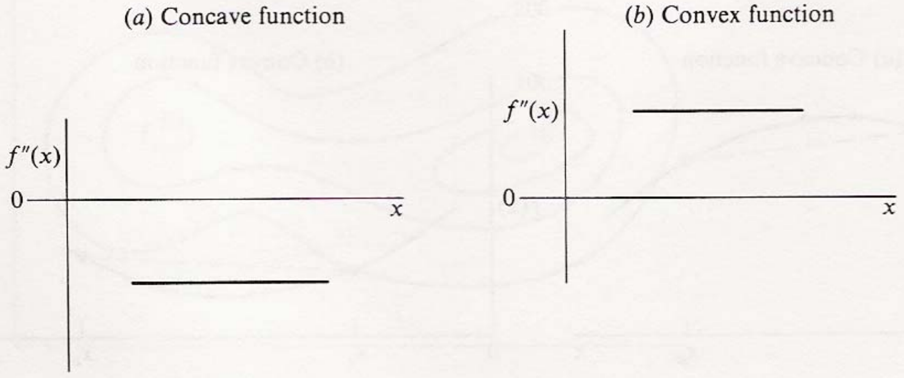

Figure 4.7 Second derivative of $f(x)$ for a quadratic function.

1. $\mathbf{H}$ is *positive definite* if and only if $\mathbf{x}^T\ \mathbf{H}\ \mathbf{x}$ is >0 for all $\mathbf{x} \neq \mathbf{0}$. convexa,
2. $\mathbf{H}$ is *negative definite* if and only if $\mathbf{x}^T\ \mathbf{H}\ \mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$. concava
3. $\mathbf{H}$ is *indefinite* if $\mathbf{x}^T\ \mathbf{H}\ \mathbf{x} < 0$ for some $\mathbf{x}$ and >0 for other $\mathbf{x}$.

Two convenient tests can be used to establish the status of $\mathbf{H}(\mathbf{x})$ for strict convexity:

(1) All diagonal elements must be positive and the determinants of all leading principal minors, $\det\{M_i(\mathbf{H})\}$, and of $\mathbf{H}(\mathbf{x})$ itself, $\det(\mathbf{H})$, are positive (>0). Keep in mind that $\mathbf{H}(\mathbf{x})$ *must be a symmetric matrix.*

Another test is

(2) All the eigenvalues of $\mathbf{H}(\mathbf{x})$ are positive (>0).

Table 4.1 Relationship between the character of $f(\mathbf{x})$ and the state of $\mathbf{H}(\mathbf{x})$

$f(\mathbf{x})$ is	$\mathbf{H}(\mathbf{x})$ is	All the eigenvalues of $\mathbf{H}(\mathbf{x})$ are	Determinants of the leading principal minors of $\mathbf{H}^*$ (Δ_i)
Strictly convex	Positive definite	>0	$\Delta_1 > 0, \Delta_2 > 0, \ldots$
Convex	Positive semi-definite	≥ 0	$\Delta_1 \geq 0, \Delta_2 \geq 0, \ldots$
Concave	Negative semi-definite	≤ 0	$\Delta_1 \leq 0, \Delta_2 \geq 0, \Delta_3 \leq 0, \ldots$ (alternating sign)
Strictly concave	Negative definite	<0	$\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0$ (alternating sign)

(1) All diagonal elements must be negative and $\det(\mathbf{H})$ and $\det\{M_i(\mathbf{H})\} > 0$ if i is even ($i = 2, 4, 6, \ldots$); $\det(\mathbf{H})$ and $\det\{M_i(\mathbf{H})\} < 0$ if i is odd ($i = 1, 3, 5, \ldots$), where M_i is the ith principal minor determinant.

(2) All the eigenvalues of $\mathbf{H}(\mathbf{x})$ are negative (<0).

Región Convexa

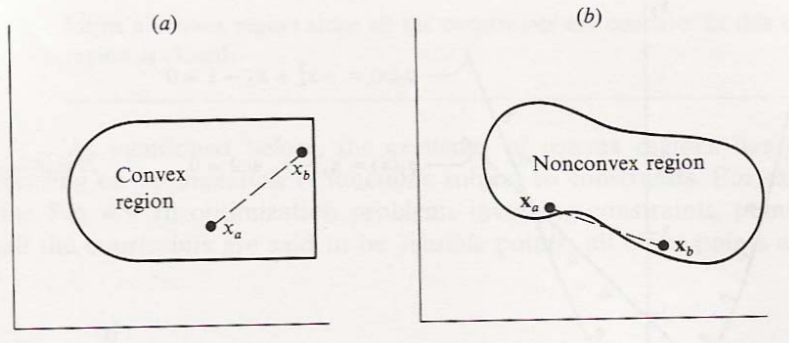

Figure 4.8 Convex and nonconvex regions.

Si una región esta completamente limitada por funciones concavas del tipo g(x) ≥ 0, entonces las funciones forman una región cerrada convexa.

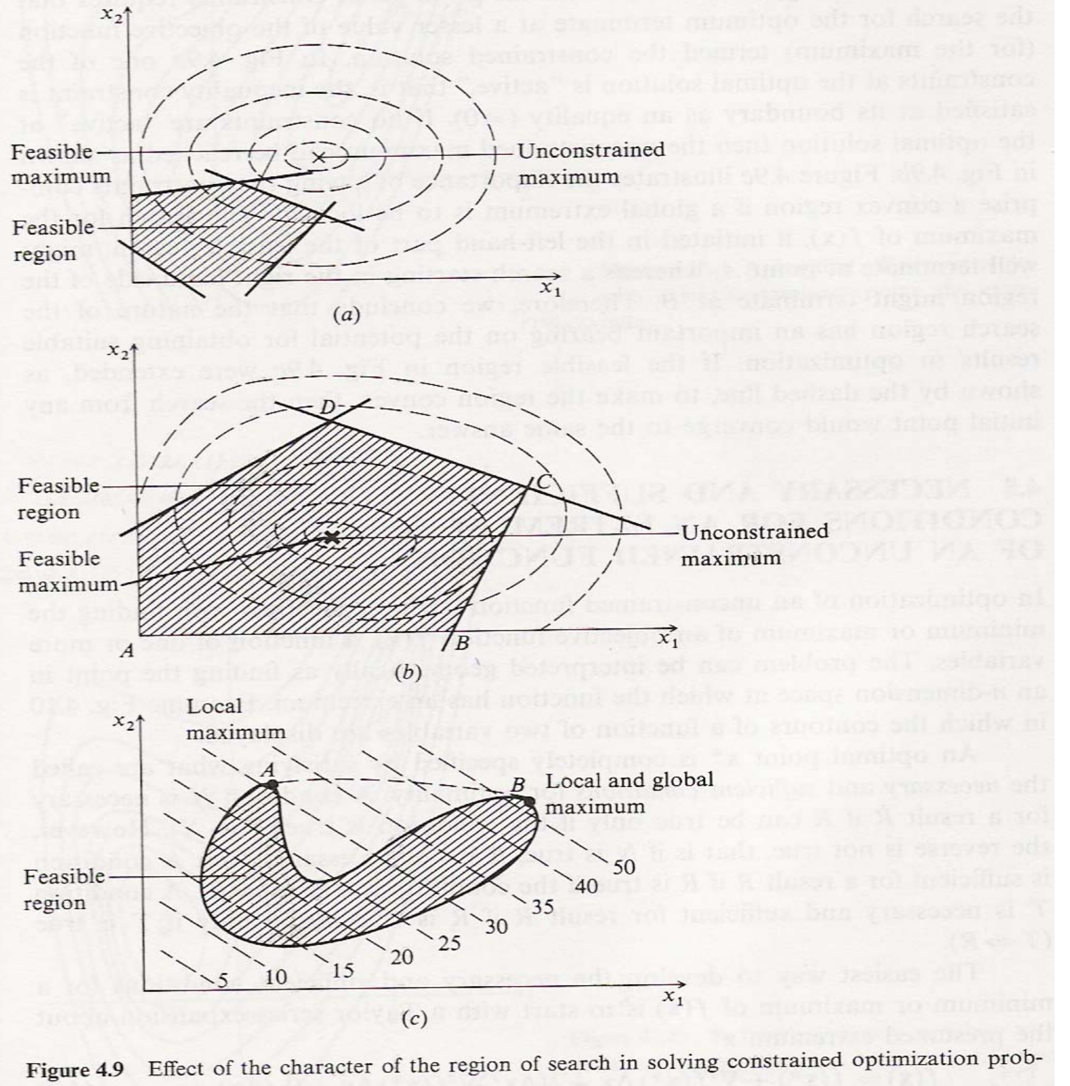

Figure 4.9 Effect of the character of the region of search in solving constrained optimization problems.

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